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Finite-Size Scaling Corrections for the Eight-Vertex Model

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Finite-size scaling corrections are calculated analytically for two of the maximal eigenvalues of the transfer matrix in the isotropic eight-vertex model. The value $c = 1$ for the conformal anomaly of the Virasoro algebra is confirmed.

KEY WORDS: Finite-size scaling; eight-vertex model; conformal invariance.

1. INTRODUCTION

Recently, de Vega and Woynarovitch[1] have shown how to derive the leading-order finite-size corrections analytically for any model that is soluble by the Bethe ansatz. The method has been applied to the ground-state energy of the *XXZ* Heisenberg chain in its critical region,[2-4] and the results have been related to critical indices of the model using conformal invariance.[5] Now the *XXZ* model is just the quantum Hamiltonian corresponding to the critical eight-vertex model in its extreme anisotropic limit. Here we set out to complement the above results by deriving similar finite-size corrections for the isotropic eight-vertex model.

2. FINITE-SIZE SCALING CORRECTIONS FOR THE EIGHT-VERTEX MODEL

To save space, we shall employ the notation of Johnson *et al.*[6] (JKM), who rederived Baxter's results[7,8] for the eight-vertex model using

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an integral equation method. The eigenvalues $T(v)$ of the transfer matrix satisfy the equation

$$T(v)\,Q(v) = \phi(v+\eta)\,Q(v-2\eta) + \phi(v-\eta)\,Q(v+2\eta) \tag{2.1}$$

where

$$\phi(v) = [\rho\theta(0)\,h(v)]^N \tag{2.2a}$$

$$Q(v) = \exp\left(-i\frac{\nu\pi v}{2K_k}\right)\prod_{j=1}^{r} h(v-v_j) \tag{2.2b}$$

$$h(v) = H(v)\,\theta(v) \tag{2.2c}$$

Here $N=2r$ is the number of vertices in a row, $H(v)$ and $\theta(v)$ are standard elliptic theta functions, and the parameters v, η, ν, K_k are as defined by JKM. The zeros v_j of $Q(v)$ are given by a set of coupled nonlinear equations, obtained by noting that the left-hand side of (2.1) vanishes when $v=v_j$ $(j=1,\dots,r)$:

$$\left[\frac{h(v_j+\eta)}{h(v_j-\eta)}\right]^N = -e^{-2\pi i\nu\eta/K_k}\prod_{j=1}^{r}\frac{h(v_j-v_i+2\eta)}{h(v_j-v_i-2\eta)}, \qquad j=1,\dots,r \tag{2.3}$$

Taking the logarithm of this equation, one obtains

$$NF_1(\phi_j) = -2\pi i l_j + 2\nu\lambda + \sum_{i=1}^{r} F_2(\phi_j-\phi_i), \qquad j=1,\dots,r \tag{2.4}$$

where the l_j are half-integers specifying the branches of the logarithm function, $\phi_j = \pi v_j/K_k$, and for the largest eigenvalues

$$|l_j - l_{j-1}| = 1, \qquad j=2,\dots,r \tag{2.5}$$

The function

$$F_p(\phi_j) = \ln\frac{h(v_j+p\eta)}{h(v_j-p\eta)} \tag{2.6}$$

is chosen so that no cuts of F_p cross the real axis, and $F_p(0) = -i\pi (p\neq 0)$. It is a "quasiperiodic" function, e.g., on the real axis it is periodic except for a linear term such that

$$F_p(n\pi) = -i(n+1)\pi \tag{2.7}$$

(see JKM, Appendix B).

In evaluating the transfer matrix eigenvalues, we shall write

$$T(v)=\phi(v+\eta)\frac{Q(v-2\eta)}{Q(v)}[1+r(v)] \tag{2.8}$$

where

$$r(v)=\frac{\phi(v-\eta)\,Q(v+2\eta)}{\phi(v+\eta)\,Q(v-2\eta)} \tag{2.9}$$

Using Eq. (A13) of JKM, one finds

$$T(v)=c^N\left(\frac{\theta(0)}{\theta(v-\eta)}\frac{H(v+\eta)}{H(2\eta)}\right)^N\frac{Q(v-2\eta)}{Q(v)}[1+r(v)] \tag{2.10}$$

Hence, using the Fourier transforms in JKM, Appendix B, one arrives at

$$\begin{aligned}\frac{1}{N}\ln[(-1)^r\,T(v)]\\ &=\frac{i\pi}{2}+\ln c+\frac{1}{N}\ln\left[\frac{Q(v-2\eta)}{Q(v)}\right]+\frac{1}{N}\ln[1+r(v)]+\frac{\alpha-\lambda}{2}\\ &\quad+4\sum_{m=1}^{\infty}\frac{\sinh[m(\lambda-\alpha)/2]\cosh\{m[\tau-(\alpha+\lambda)/2]\}\sinh[m(\lambda-\tau)]}{m\sinh(2m\tau)}\end{aligned} \tag{2.11}$$

Now let us start again from Eq. (2.4). Following de Vega and Woynarovitch,[1] we define the function

$$Z_N(\phi)=\frac{-1}{2\pi i}\left[F_1(\phi)-\frac{2v\lambda}{N}-\frac{1}{N}\sum_{i=1}^{r}F_2(\phi-\phi_i)\right] \tag{2.12}$$

and its derivative

$$R_N(\phi)=dZ_N(\phi)/d\phi \tag{2.13}$$

such that at the roots ϕ_j

$$Z_N(\phi_j)=\frac{l_j}{N} \tag{2.14}$$

When N goes to infinity, the ϕ_j tend to a continuous distribution on the real axis with density $NR_N(\phi)$, and differentiating Eq. (2.12), one obtains a linear integral equation

$$R_\infty(\phi)=\frac{-F_1'(\phi)}{2\pi i}+\frac{1}{2\pi i}\int_{-\pi}^{\pi}d\phi'\,R_\infty(\phi')\,F_2'(\phi-\phi') \tag{2.15}$$

Similarly, one finds in the thermodynamic limit that

$$\frac{1}{N}\ln\frac{Q(v-2\eta)}{Q(v)} = -i\pi - \frac{v\lambda}{N} - \frac{1}{N}\sum_{j=1}^{r} F_1(i(\alpha-\lambda)-\phi_j)$$

$$\underset{N\to\infty}{\simeq} -i\pi - \int_{-\pi}^{\pi} d\phi'\, R_\infty(\phi')\, F_1(i(\alpha-\lambda)-\phi') \quad (2.16)$$

and

$$r(v) = \exp[2\pi i N Z_N(i\alpha)] \quad (2.17)$$

Now Eq. (2.15) can be solved by a Fourier series expansion to give

$$R_\infty(\phi) = \sum_{m=-\infty}^{+\infty} \frac{e^{im\phi}}{4\pi\cosh m\lambda} = \frac{K_1}{2\pi^2}\, dn\left(\frac{K_1\phi}{\pi}, k_1\right) \quad (2.18)$$

where $K_1 \equiv K_1(k_1)$ and $dn(z, k_1)$ are elliptic functions of modulus k_1, with $K_1'(k_1)/K_1(k_1) = \lambda/\pi$. Substituting in Eq. (2.16) and using JKM, Eq. (B13), we find

$$\frac{1}{N}\ln\frac{Q(v-2\eta)}{Q(v)} = \frac{1}{2}(\lambda-\alpha-i\pi) + \sum_{m=1}^{\infty} \frac{\sinh m(\tau-\lambda)\sinh m(\lambda-\alpha)}{m\sinh m\tau\cosh m\lambda} \quad (2.19)$$

and from Eq. (2.11)

$$\frac{1}{N}\ln\Lambda_0 \underset{N\to\infty}{\simeq} \ln c + 2\sum_{m=1}^{\infty} \frac{\sinh^2 m(\tau-\lambda)[\cosh m\lambda - \cosh m\alpha]}{m\sinh 2m\tau\cosh m\lambda} \quad (2.20)$$

This agrees with Baxter's result[7,8] for the bulk limit of the maximum eigenvalue.

De Vega and Woynarovitch[1] show that one can derive similar integral equations valid for any N. The definitions (2.12) and (2.13) give

$$R_N(\phi) = \frac{-1}{2\pi i} F_1'(\phi) + \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\phi'\, R_N(\phi')\, F_2'(\phi-\phi')$$

$$+ \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\phi'\, F_2'(\phi-\phi')\left[\frac{1}{N}\sum_j \delta(\phi'-\phi_j) - R_N(\phi')\right] \quad (2.21)$$

and hence

$$R_N(\phi) - R_\infty(\phi)$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\phi'\, F_2'(\phi-\phi')[R_N(\phi') - R_\infty(\phi')]$$

$$+ \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\phi'\, F_2'(\phi-\phi')\left[\frac{1}{N}\sum_j \delta(\phi'-\phi_j) - R_N(\phi')\right] \quad (2.22)$$

This can be manipulated to give

$$R_N(\phi) - R_\infty(\phi) = \frac{-1}{\pi} \int_{-\pi}^{\pi} d\phi' \, P(\phi - \phi') \left[\frac{1}{N} \sum_j \delta(\phi' - \phi_j) - R_N(\phi') \right] \tag{2.23}$$

where

$$P(\phi) = \sum_{m=-\infty}^{+\infty} \tilde{P}_m e^{im\phi} \tag{2.24}$$

with

$$\tilde{P}_m = \begin{cases} \dfrac{1}{4}, & m = 0 \\ \dfrac{\sinh m(\tau - 2\lambda)}{4 \cosh m\lambda \sinh m(\tau - \lambda)}, & m \neq 0 \end{cases} \tag{2.25}$$

For the eigenvalue, if we define

$$f_N(v) = \frac{1}{N} \ln[(-1)^r \, T(v)] \tag{2.26}$$

and use (2.11) and (2.16) to define the limiting value $f_\infty(v)$, we have

$$\begin{aligned} f_N(v) - f_\infty(v) = {} & \frac{1}{N} \ln[1 + r(v)] - \frac{v\lambda}{N} - \int_{-\pi}^{\pi} d\phi' \, F_1(i(\alpha - \lambda) - \phi') \\ & \times \left[\frac{1}{N} \sum_j \delta(\phi' - \phi_j) - R_N(\phi') \right] \\ & - \int_{-\pi}^{\pi} d\phi' \, F_1(i(\alpha - \lambda) - \phi')[R_N(\phi') - R_\infty(\phi')] \end{aligned} \tag{2.27}$$

Now the analogue of the relation (2.35) of De Vega and Woynarovitch is

$$\int_{-\pi}^{\pi} \frac{d\phi'}{\pi} F_1'(\phi') \, P(\phi - \phi') = 2\pi i R_\infty(\phi) + F_1'(\phi) \tag{2.28}$$

which can be integrated to give

$$\begin{aligned} & \int_{-\pi}^{\pi} \frac{d\phi'}{\pi} F_1(\phi' - \phi_0) \, P(\phi - \phi') \\ & = iam\left(\frac{K_1}{\pi}(\phi + \phi_0), k_1\right) + \frac{i\pi}{2} + F_1(\phi + \phi_0) + 2i \int_{-\pi}^{\phi - \pi} d\phi' \, P(\phi') \end{aligned} \tag{2.29}$$

for $|\lambda - \tau| + |\text{Im } \phi_0| < \tau$.

Substituting this result into Eq. (2.27), one obtains:

$$f_N(v)-f_\infty(v)=\frac{1}{N}\ln[1+r(v)]-\frac{v\lambda}{N}$$
$$+\int_{-\pi}^{\pi} d\phi'\left[iam\left(\frac{K_1}{\pi}(i(\alpha-\lambda)-\phi'),k_1\right)\right.$$
$$\left.+\frac{i\pi}{2}+2i\int_{-\pi}^{-\phi'-\pi}d\phi''\,P(\phi'')\right]\left[\frac{1}{N}\sum_j\delta(\phi'-\phi_j)-R_N(\phi')\right] \tag{2.30}$$

3. THE CRITICAL REGION

At this point, let us specialize to the critical region of the eight-vertex model, in which case

$$K\to\infty,\qquad K'\to\pi/2 \tag{3.1}$$

and it becomes convenient to define new, rescaled variables corresponding to those used by De Vega and Woynarovitch[1]:

$$\mu=2K\phi/\pi=2v \tag{3.2a}$$

$$\gamma=-4i\eta \tag{3.2b}$$

$$z_N(\mu)=Z_N(\phi) \tag{3.2c}$$

$$dz_N(\mu)/d\mu\equiv\sigma_N(\mu)=(\pi/2K)\,R_N(\phi) \tag{3.2d}$$

$$-i\phi(\mu,p\lambda/2)=F_p(\phi) \tag{3.2e}$$

$$p(\mu)=(\pi/2K)\,P(\phi) \tag{3.2f}$$

We shall also restrict our attention to the isotropic case, $v=0$. Then our final equations for finite N translate to

$$\sigma_N(\mu)-\sigma_\infty(\mu)=\frac{-1}{\pi}\int_{-\infty}^{\infty}d\mu'\,p(\mu-\mu')\left[\frac{1}{N}\sum_j\delta(\mu'-\mu_j)-\sigma_N(\mu')\right] \tag{3.3}$$

and

$$f_N(0)-f_\infty(0)=\frac{1}{N}\ln[1+r(0)]+2\int_{-\infty}^{\infty}d\mu'\,\operatorname{arctanh}(e^{-\pi\mu'/\gamma})$$
$$\times\left[\frac{1}{N}\sum_j\delta(\mu'-\mu_j)-\sigma_N(\mu')\right] \tag{3.4}$$

using the fact that

$$K_1/K \to 2\pi/\gamma$$

in the limit (3.1), and that

$$\int_{-\pi}^{-\phi'-\pi} d\phi''\, P(\phi'')$$

is $O(1/K)$, and vanishes in the same limit.

3.1. Case 1: The Eigenvalue Λ_0

This case was discussed by JKM. The half-integers I_j are given by

$$I_j = (j - 1/2), \qquad j = 1,..., r \tag{3.5}$$

and $\nu = 0$. The roots are symmetrically distributed about $\phi = 0$, with $Z_N(-\pi) = 0$, $Z_N(0) = 1/4$, and $Z_N(\pi) = 1/2$. In the critical region, we have

$$z_N(\mu_i) = (i - 1/2)/N, \qquad i = 1,..., r \tag{3.6}$$

at the root positions. We have to evaluate expressions of the form

$$I_N = \int_{-\infty}^{\infty} d\mu'\, f(\mu') \left[\frac{1}{N} \sum_{i=1}^{r} \delta(\mu' - \mu_i) - \sigma_N(\mu') \right] \tag{3.7}$$

$$= \int_0^{1/2} dz_N\, f(\mu'(z_N)) \left[\frac{1}{N} \sum_{i=1}^{r} \delta(z_N - z_N^i) - 1 \right] \tag{3.8}$$

The sum over delta functions can be written as a Fourier series as follows:

$$\frac{1}{N} \sum_{i=1}^{r} \delta(z_N - z_N^i) = \sum_{m=-\infty}^{+\infty} (-1)^m\, e^{2\pi i m N z_N} \tag{3.9}$$

whence one obtains

$$I_N = \sum_{\substack{m=-\infty \\ (m \neq 0)}}^{\infty} (-1)^m \int_{-\infty}^{\infty} d\mu'\, f(\mu')\, \sigma_N(\mu')\, e^{2\pi i m N z_N(\mu')} \tag{3.10}$$

The asymptotic root density, from Eq. (2.18), is given by

$$\sigma_\infty(\mu) = \frac{1}{2\gamma \cosh(\pi\mu/\gamma)} \tag{3.11}$$

and hence

$$z_\infty(\mu) = \frac{1}{\pi} \arctan(e^{\pi\mu/\gamma}) \tag{3.12}$$

Substituting the resummation (3.10) into Eq. (3.4), and replacing $z_N(\mu')$ by $z_\infty(\mu')$, we find

$$f_N(0) - f_\infty(0) \cong \frac{1}{N} \ln[1 + r(0)] + 4 \sum_{m=1}^{\infty} (-1)^{1+m} \int_0^{1/4} dz' \cos(2\pi m N z') \ln \tan \pi z' \tag{3.13}$$

where $z' = z - 1/4$, provided $N/2$ is even. But

$$r(0) = \exp[2\pi i N z_N(0)] = e^{i\pi N/2} \tag{3.14}$$

so that

$$\frac{1}{N} \ln[1 + r(0)] = \frac{1}{N} \ln 2 \tag{3.15}$$

for $N/2$ even; while in the Appendix it is shown that

$$\sum_{m=1}^{\infty} (-1)^{1+m} \int_0^{1/4} dz' \cos(2\pi m N z') \ln \tan(\pi z') = -\frac{1}{4N} \ln 2 + \frac{\pi}{24N^2} + O(N^{-3}) \tag{3.16}$$

Hence one obtains the final result

$$f_N(0) - f_\infty(0) \cong \pi/6N^2 \tag{3.17}$$

For the case $N/2$ *odd*, one finds that the term $(1/N) \ln[1 + r(0)]$ diverges to negative infinity as one approaches the critical region, while the term

$$(1/N) \ln[Q(-2\eta)/Q(0)]$$

diverges to positive infinity. This merely reflects the fact that the point $v = 0$ is a root, and $Q(0)$ vanishes. We have not yet been able to derive an explicit result for this case.

It remains to show that any corrections due to the difference

$$[z_N(\mu') - z_\infty(\mu')]$$

are negligible in comparison to the leading-order term, Eq. (3.17), as $N \to \infty$. In a previous paper[3] we gave arguments to this effect for the case of the XXZ model; but recent work by Woynarovitch and Eckle[9] has shown that these arguments are not strictly correct, although the conclusion still holds. For our present purposes, we shall simply *assume* the corrections are negligible, and refer to Woynarovitch and Eckle[9] for a discussion of how to estimate the nonleading correction terms.

3.2. Case 2: The Eigenvalue Λ_1

There is another eigenvalue Λ_1 equal in magnitude and opposite in sign to Λ_0 in the thermodynamic limit, as discussed by Baxter[8] and JKM. For this case the half-integers are given by

$$I_j = (j + 1/2), \qquad j = 1, ..., r \tag{3.18}$$

except for one root at $\phi = \pi$. We have

$$Z_N(-\pi) = \frac{1}{2N}, \qquad Z_N(0) = \frac{1}{4} + \frac{1}{2N}, \qquad Z_N(\pi) = \frac{1}{2} + \frac{1}{2N}$$

The asymptotic root density and eigenvalue in the thermodynamic limit $N \to \infty$ are the same as for Λ_0, so we may proceed directly to look at the finite-size corrections. In this case,

$$I_N = \int_{-\infty}^{\infty} d\mu' \, f(\mu') \left[\frac{1}{N} \sum_{i=1}^{r} \delta(\mu' - \mu_i) - \sigma_N(\mu') \right] \tag{3.19}$$

$$= \int_0^{1/2} dz'_N \, f(\mu'(z'_N)) \left[\frac{1}{N} \sum_{i=1}^{r} \delta(z'_N - z'^i_N) - 1 \right] \tag{3.20}$$

where $z'_N = z_N - 1/(2N)$, $z'^i_N = i/N$, and after Fourier expanding, we have

$$I_N = \sum_{\substack{m=-\infty \\ (m \neq 0)}}^{+\infty} \int_{-\infty}^{\infty} d\mu' \, f(\mu') \, \sigma_N(\mu') \exp[2\pi i m N z'_N(\mu')] \tag{3.22}$$

Substituting this resummation into Eq. (2.30), and replacing $z'_N(\mu')$ by $z'_\infty(\mu')$, we find

$$f_N(0) - f_\infty(0) \cong \frac{1}{N} \ln[1 + r(0)] - 4 \sum_{m=1}^{\infty} (-1)^{mN/2} \times \int_0^{1/4} dz'' \cos(2\pi m N z'') \ln \tan(\pi z'') + iC \tag{3.23}$$

where $z'' = z' - 1/4$, and C is a constant discussed below. Now

$$r(0) = -e^{i\pi N/2} \tag{3.24}$$

so the natural choice in this case is to take $N/2$ *odd*, when

$$f_N(0) - f_\infty(0) \cong \frac{1}{N} \ln 2 - 4 \sum_{m=1}^{\infty} (-1)^m \times \int_0^{1/4} dz'' \cos(2\pi m N z'') \ln \tan(\pi z'') + ic \tag{3.25}$$

which, using the results of the Appendix, reduces to

$$f_N(0) - f_\infty(0) \cong -\pi/3N^2 + iC \tag{3.26}$$

This result is subject to the same assumption as in case 1, namely that corrections due to $[z'_N(\mu) - z'_\infty(\mu)]$ are smaller than $O(N^{-2})$.

The constant C, due to an annoying technicality, cannot be directly computed in the critical region. The origin of the difficulty lies in the linear, or nonperiodic, terms appearing in the Fourier expansions of some of the elliptic and F_p functions. These are not properly accounted for in taking the critical limit, Eqs. (3.1)–(3.4). By going back to Eq. (2.30) and looking at the linear terms more carefully, one may correctly obtain $C = -\pi/N$ for this case. Thus

$$\ln(\Lambda_1/\Lambda_0) \underset{N \to \infty}{\sim} -i\pi$$

i.e., Λ_1 and Λ_0 have opposite signs, in agreement with the results of Baxter[8] and JKM.

4. CONCLUSIONS

Assuming the hypothesis of conformal invariance in the critical region, it has been shown by Blöte *et al.*[10] and Affleck[11] that the finite-size scaling amplitude for the maximum eigenvalue of the transfer matrix is given by

$$f_N(0) - f_\infty(0) \underset{N \to \infty}{\sim} \pi c/6N^2 \tag{4.1}$$

for periodic boundary conditions, where c is the conformal anomaly or central charge of the Viasoro algebra. Our result (3.17) implies $c = 1$ for the eight-vertex model, in agreement with that obtained for the XXZ Hamiltonian.[2,3] The amplitude for the eigenvalue Λ_1 appears to be the same as that for the "kink mass" in the XXZ model, which we argued[3] is connected with the exponent $\eta = 1/4$ by conformal invariance.

Some numerical studies have recently been done on the eight-vertex model,[12] which confirm the result (3.17).

It should certainly be possible to derive analytic expressions for the finite-size scaling behavior of other eigenvalues of the transfer matrix, involving complex roots, but this is a more difficult algebraic task.

APPENDIX

We want to find the asymptotic behavior of the integral

$$I_1 = \int_0^{1/4} dz \cos(4\pi m z) \ln \tan \pi z \tag{A1}$$

for large m. Integrating by parts, one finds

$$I_1 = \frac{-1}{2m} \int_0^{1/4} dz \frac{\sin(4\pi m z)}{\sin(2\pi z)} = \frac{-1}{2\pi m}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{m-1}}{2m-1}\right) \tag{A2}$$

The behavior of this series partial sum may be found using the Euler summation formula for alternating series,[13]

$$\begin{aligned} f(1) - f(2) + f(3) - \cdots + (-1)^{x-1} f(x) \\ = \text{const} + (-1)^{x-1}\left[\frac{1}{2} f(x) + \frac{2^2-1}{2!} B_1 f'(x) \cdots\right] \end{aligned} \tag{A3}$$

with $f(x) = 1/(2x-1)$ and $x = n$:

$$-\frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1} = \text{const} + \frac{(-1)^{n-1}}{2(2n-1)} + O(n^{-2}) \tag{A4}$$

But the constant term is known,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \tag{A5}$$

and so one obtains the result

$$I_1 = \frac{-1}{8m} + \frac{(-1)^m}{8\pi m^2} + O(m^{-3}) \tag{A6}$$

Now consider the expression in Eq. (47) of the text:

$$I_2 = \sum_{m=1}^{\infty} (-1)^{1+m} \int_0^{1/4} dz \cos(2\pi m N z) \ln \tan \pi z \tag{A7}$$

$$= \frac{1}{4N} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} - \frac{1}{2\pi N^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} + O(N^{-3}) \tag{A8}$$

using (A6), with $N/2$ even; hence the result quoted in Eq. (47). For the case $N/2$ odd, we have

$$I_2 = \frac{1}{4N} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} - \frac{1}{2\pi N^2} \sum_{m=1}^{\infty} \frac{1}{m^2} + O(N^{-3}) \tag{A9}$$

which gives Eq. (57).

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